Announcements

1) Math Club talk, tomorrow
3-4, CB 2046

2) Last homework now up
on CTools, due Tuesday
next week.

Important Result for Derivatives

If $\underline{f(x) = c}$ for all x in $[a,b]$, then

$\underline{f'(x) = 0}$ for all x in $[a,b]$.

Note $$f'(x) = \lim_{y \to x} \frac{f(x) - f(y)}{x - y}$$

$$= \lim_{y \to x} \frac{c - c}{x - y} = 0$$

for all x in $[a,b]$

Theorem: (Intermediate value)

Let f be continuous on an interval I. (range in $\mathbb{R}$)

If $x, y \in I$ and $f(x) < f(y)$, then for all α with $f(x) < \alpha < f(y)$, $\exists\ z \in I$ with

$$\boxed{f(z) = \alpha}$$

proof: I is an interval, so if $x, y \in I$, either $[x,y]$ or $[y,x]$ is contained in I, depending on whether $x > y$ or $y > x$. Assume, without loss of generality, that $y > x$.

Given $[x,y]$, which is a connected set, since f is continuous, $f([x,y])$ is connected

We showed that all connected subsets of $\mathbb{R}$ are intervals, so

$f([x,y])$ is an interval.

If $f(x) < \alpha < f(y)$, then

$\alpha \in f([x,y])$ since $f([x,y])$ is an interval. Therefore,

$\exists$ $z \in [x,y] \subseteq I$ with

$f(z) = \alpha$. $\square$

Definition: (local max/min)

Let $f: \mathbb{R} \to \mathbb{R}$. A point $x \in \mathbb{R}$ is called a local maximum of f if $\exists\ \varepsilon > 0$ such that $\forall\ y \in (x-\varepsilon, x+\varepsilon)$, $y \neq x$, $f(y) < f(x)$

Similarly, x is a local minimum if $f(y) > f(x)$ $\forall\ y \in (x-\varepsilon, x+\varepsilon)$

Remark: You can replace the domain "$\mathbb{R}$" of f with any metric space and the definition holds. The range space is more problematic.

Lemma: Let $f: [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$.

Then if f has a local maximum or local minimum at $s \in [a,b]$, then $\boxed{f'(s) = 0}$

proof: Assume, without loss of generality, that s is a local maximum of f

Then $\exists\ \varepsilon > 0$ with

$$f(y) < f(s) \quad \forall\ y \in (s-\varepsilon, s+\varepsilon).$$

Suppose $y > s$, $y \in (s-\varepsilon, s+\varepsilon)$.

Then $\dfrac{f(y)-f(s)}{y-s} < 0$

since $y > s$ (denominator positive)
and $f(y) < f(s)$ (numerator negative).

Suppose $y < s$, $y \in (s-\varepsilon, s+\varepsilon)$

Then $\dfrac{f(y)-f(s)}{y-s} > 0$.

This shows that if you calculate

$$f'(s) = \lim_{y \to s} \frac{f(y) - f(s)}{y - s},$$

the numbers in the limit are always negative if $y > s$ and positive if $y < s$.

Since the limit exists, we then have $f'(s) = 0$

(the only number that is both a limit of positive and negative numbers)

The proof for local minima is similar. $\square$

Theorem: (Rolle) If $f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$. If $f(a)=f(b)$, then $\exists\ c\in(a,b)$ with

$$\boxed{f'(c)=0}.$$

proof: 2 cases:

1) $\exists\ a\le x<y\le b$ with f constant on $[x,y]$.

Then $\forall\ s \in [x,y]$,

$f'(s) = 0$ since f is constant, so let c be any point in (x,y).

2) There is no subinterval of $[a,b]$ on which f is constant

Since f is differentiable on $[a,b]$, f is continuous on $[a,b]$

The interval $[a,b]$ is compact, so f attains its maximum M and minimum m on $[a,b]$.

Since f is nonconstant, $m < M$. Also since $f(a)=f(b)$, we either have $m \neq f(a)$ or $M \neq f(a)$. Without loss of generality, suppose $M \neq f(a)$.

Since $\exists$ $c \in (a,b)$ with $f(c) = M$, $f(c) > f(y)$ for all y in some interval $(c-\varepsilon, c+\varepsilon)$ since f is nonconstant on any subinterval and $f(c) \geq f(s)$ for all $s \in [a,b]$. Therefore, $x = c$ is a local maximum for f, and by the lemma,

$$f'(c) = 0. \qquad \square$$

Theorem: (Mean value) If f is differentiable on $[a,b]$ $(f: [a,b] \to \mathbb{R})$, then $\exists$ $c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof. Define, for $x \in [a,b]$

$$g(x) = f(x) - x\left(\frac{f(b)-f(a)}{b-a}\right).$$

Check: $g(a) = g(b)$!

Then g satisfies the hypotheses of Rolle's Theorem, so $\exists$ a point $c \in (a,b)$ with

$g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a},$$

so $g'(c)=0$ is the same as

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \square$$

Proposition: (functions that differ by a constant)

If $f, g : [a,b] \to \mathbb{R}$ and are differentiable on $[a,b]$, then if

$$\underline{f'(x) = g'(x)} \quad \forall x \in [a,b],$$

$\exists\, c \in \mathbb{R}$ with

$$\underline{f(x) = g(x) + c} \quad \forall x \in [a,b]$$