

Announcements

1) Math Club talk, tomorrow

3-4, C13 2046

2) Last homework now up

on CTools, due Tuesday
next week.

Important Result for Derivatives

If $f(x) = c$ for all x
in $[a, b]$, then

$f'(x) = 0$ for all x in
 $[a, b]$.

Note $f'(x) = \lim_{y \rightarrow x} \frac{f(x) - f(y)}{x - y}$

$$= \lim_{y \rightarrow x} \frac{c - c}{x - y} = 0$$

for all x in $[a, b]$

Theorem: (Intermediate value)

Let f be continuous

on an interval I . (range in \mathbb{R})

If $x, y \in I$ and

$f(x) < f(y)$, then for

all α with $f(x) < \alpha < f(y)$,

$\exists z \in I$ with

$$f(z) = \alpha$$

PROOF: \overline{I} is an interval,

so if $x, y \in I$, either

$[x, y]$ or $[y, x]$ is contained

in I , depending on whether

$x > y$ or $y > x$. Assume,

without loss of generality, that

$y > x$.

Given $[x, y]$, which is

a connected set, since

f is continuous,

$f([x, y])$ is connected

We showed that all connected subsets of \mathbb{R} are intervals, so

$f([x, y])$ is an interval.

If $f(x) < z < f(y)$, then

$z \in f([x, y])$ since $f([x, y])$ is an interval. Therefore,

$\exists z \in [x, y] \subseteq I$ with

$$f(z) = z.$$



Definition: (local max/min)

Let $f: \mathbb{R} \rightarrow \mathbb{R}$. A point

$x \in \mathbb{R}$ is called a local maximum

of f if $\exists \varepsilon > 0$ such that

$\forall y \in (x - \varepsilon, x + \varepsilon), y \neq x,$

$$f(y) < f(x)$$

Similarly, x is a local minimum

if $f(y) > f(x) \quad \forall y \in (x - \varepsilon, x + \varepsilon)$

Remark: You can replace

the domain " \mathbb{R} " of f
with any metric space
and the definition holds.

The range space is
more problematic.

Lemma: Let $f: [a, b] \rightarrow \mathbb{R}$

be differentiable on $[a, b]$.

Then if f has a local maximum or local minimum

at $s \in [a, b]$, then $f'(s) = 0$

Proof: Assume, without loss of

generality, that s is a local maximum of f

Then $\exists \varepsilon > 0$ with

$$f(y) < f(s) \quad \forall y \in (s - \varepsilon, s + \varepsilon).$$

Suppose $y > s$, $y \in (s - \varepsilon, s + \varepsilon)$.

Then $\frac{f(y) - f(s)}{y - s} < 0$

since $y > s$ (denominator positive)

and $f(y) < f(s)$ (numerator negative).

Suppose $y < s$, $y \in (s - \varepsilon, s + \varepsilon)$

Then $\frac{f(y) - f(s)}{y - s} > 0$.

This shows that if you calculate

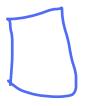
$$f'(s) = \lim_{y \rightarrow s} \frac{f(y) - f(s)}{y - s},$$

the numbers in the limit are always negative if $y > s$ and positive if $y < s$.

Since the limit exists, we

then have $f'(s) = 0$

(the only number that is both a limit of positive and negative numbers)

The proof for local
minima is similar. 

Theorem: (Rolle) If

$f: [a,b] \rightarrow \mathbb{R}$ is differentiable

on $[a,b]$. If $f(a) = f(b)$,

then $\exists c \in (a,b)$ with

$$f'(c) = 0.$$

Proof: 2 cases:

1) $\exists a \leq x < y \leq b$ with

f constant on $[x,y]$.

Then $\forall s \in [x, y]$,

$f'(s) = 0$ since f is constant, so let c be any point in (x, y) .

2) There is no subinterval of $[a, b]$ on which f is constant

Since f is differentiable on $[a, b]$, f is continuous on $[a, b]$

The interval $[a, b]$ is compact,
so f attains its maximum M
and minimum m on $[a, b]$.

Since f is nonconstant,

$m < M$. Also since

$f(a) = f(b)$, we either

have $m \neq f(a)$ or $M \neq f(a)$.

Without loss of generality,

Suppose $M \neq f(a)$.

Since $\exists c \in (a, b)$ with
 $f(c) = M$, $f(c) > f(y)$
 for all y in some interval
 $(c - \varepsilon, c + \varepsilon)$ since f is
 nonconstant on any subinterval
 and $f(c) \geq f(s)$ for all
 $s \in [a, b]$. Therefore,
 $x = c$ is a local maximum
 for f , and by the lemma,
 $f'(c) = 0$. □

Theorem: (Mean value) If

f is differentiable on

$[a, b]$ ($f: [a, b] \rightarrow \mathbb{R}$),

then $\exists c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: Define, for $x \in [a, b]$

$$g(x) = f(x) - x \left(\frac{f(b) - f(a)}{b - a} \right).$$

Check: $g(a) = g(b)$!

Then g satisfies the hypotheses of Rolle's Theorem, so $\exists a$ point $c \in (a, b)$ with

$g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so $g'(c) = 0$ is the same as

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \square$$

Proposition: (functions that differ by a constant)

If $f, g: [a, b] \rightarrow \mathbb{R}$

and are differentiable

on $[a, b]$, then if

$$\underline{f'(x) = g'(x)} \quad \forall x \in [a, b],$$

$\exists c \in \mathbb{R}$ with

$$\underline{f(x) = g(x) + c} \quad \forall x \in [a, b]$$